

# Introduction to Knot Theory

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## 1 Introduction

Knot theory is the study of mathematical knots. The basic question one asks in knot theory is, given two knots how to know if they are the same knot or not. The first mathematical study of knots was done by Alexander T. Vandermonde which included the topological properties of knots. A conjecture by Lord Kelvin that atoms were knots in the aether led to a renewed interest in the studies of knots. This conjecture motivated Peter Tait, a physicist, to create the first knot table in which he described and classified all knots upto 10 crossings. In 20th century, Max Dehn and J. W. Alexander, two topologist started studying knots from a knot group point of view and the first great knot invariant was discovered, the Alexander polynomial. The introduction of hyperbolic geometry in the study of knot by William Thurston in 1980's helped in defining many new knot invariants.

The aim of this project was to get a understanding of basic concepts in knot theory and, learn some techniques to construct and study some knot invariants, namely the Jones polynomial and the Alexander polynomial.

## 2 Basic invariants and examples

### Definitions and terminologies

**Definition 2.1.** *A subset  $L$  of  $S^3$  or  $\mathbb{R}^3$  is called a link of  $m$  components if it is a union of  $m$  disjoint, piecewise linear, simple closed curves. A link which has only one component is called a knot.*

We say a link is piecewise linear if it is made up of a finite number of straight line segments. We insist on a finite number of line segments so as to avoid a link with infinitely many kinks, i.e., to avoid a wild link of the kind shown below.

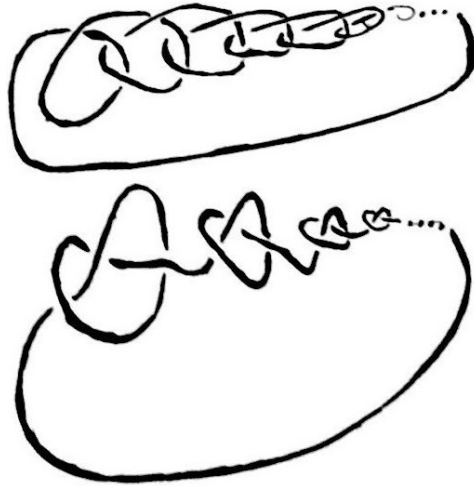


Figure 1: Wild knots

**Definition 2.2.** Links  $L_1$  and  $L_2$  in  $S^3$  are said to be equivalent if there exists an orientation preserving piecewise linear homeomorphism  $h : S^3 \rightarrow S^3$  such that  $h(L_1) = L_2$ .

A map  $h$  is said to be piecewise linear if after subdividing each of the simplices in both the  $S^3$  (i.e. the domain and range are  $S^3$  here) into many smaller simplices,  $h$  maps simplices in the domain to simplices in the co-domain in a linear way. Also  $h$  should be affine isotopic to identity i.e.  $\exists h_t : S^3 \rightarrow S^3$  such that  $h_0 = 1$  and  $h_1 = h$  and  $t \in [0, 1]$ . For each  $t$   $(x, t) \mapsto (h_t x, t)$  is a piecewise linear homeomorphism. This means that the whole  $S^3$  is continuously distorted using the homeomorphism  $h_t$  at time  $t$  to move from  $L_1$  to  $L_2$ . Equivalent links will hence forth be treated as the same link. A method of going from one link to an equivalent link is the following: find a planar triangle that intersects  $L$  in one of its sides and replace this side with the other two sides. If two links are equivalent then we can go from one link to the other by a finite number of such moves or their inverses. See Figure 2 for such triangle moves.

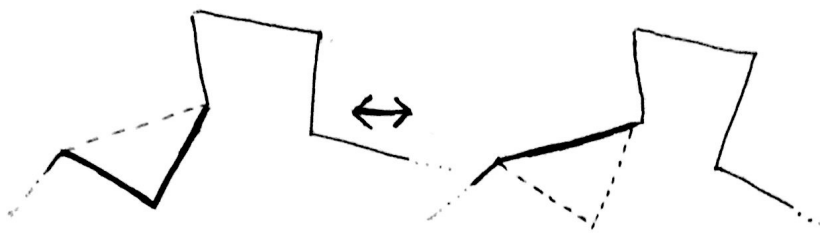


Figure 2: Triangle moves

**Definition 2.3.** Given a link  $L$ , we can make a link diagram  $D$  as follows: take a projection  $p : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  which projects each line segment of  $L$  to a line segment in  $\mathbb{R}^2$  such that projections of any two segments intersect in at most one point and the point of intersection is not the end point of the segment if they are disjoint in  $L$ . Furthermore, no point in  $\mathbb{R}^2$  belongs to the projections of three or more segments. Along with this projection when we include “over and under” information at the crossings, we get a link diagram.

There are local changes that can be done on a knot diagram called the Reidemeister moves. They are of three types as shown in the Figure 3.

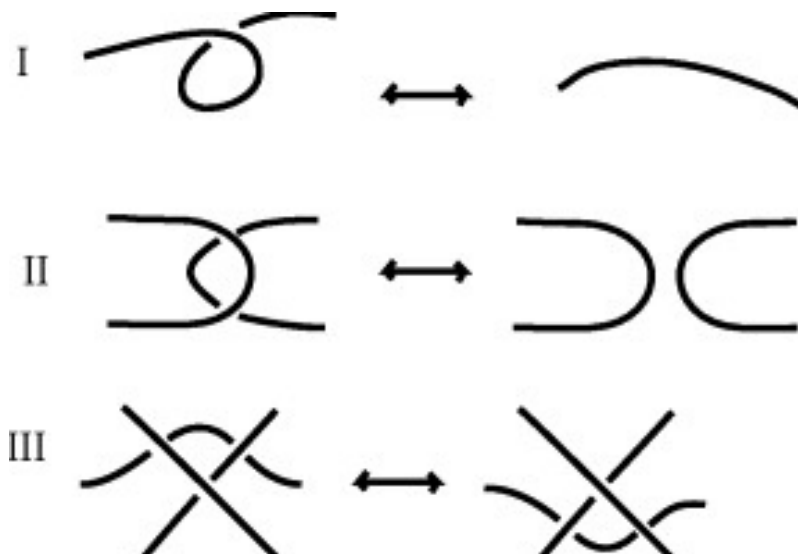


Figure 3: Reidemeister moves

**Theorem 2.4.** *Two link diagrams represent the same link iff they are related to each other by a sequence of Reidemeister moves and an orientation preserving homeomorphism of the plane.*

**Definition 2.5.** *If the diagrams of links are related to each other only by moves of type II and III, then they are said to be regular isotopic.*

The Reidemeister moves are in fact the triangle moves as shown below.

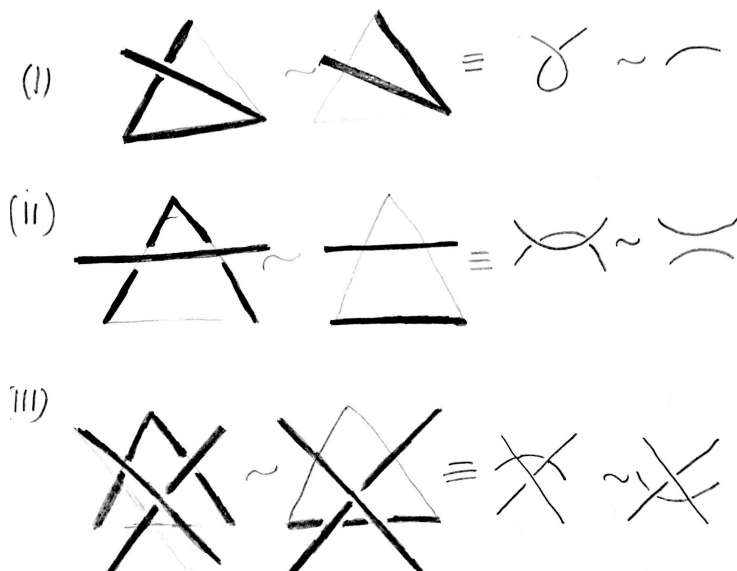


Figure 4: Reidemeister moves

**Definition 2.6.** *Reverse of an oriented knot  $K$ , denoted by  $rK$ , is the the same knot with the opposite orientation. Obverse or Reflection of a link  $L$  is a link  $\rho(L)$ , where  $\rho : S^3 \rightarrow S^3$  is an orientation reversing piecewise linear homeomorphism.*

Reverse and obverse of a trefoil knot is shown in the figure 5 below.



Figure 5: Oriented knot, its reverse and its reflection

A crossing of a oriented diagram is assigned a sign as shown in the Figure 6 .

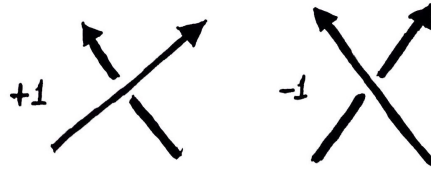


Figure 6: sign of a crossing

**Definition 2.7.** The writhe  $w(D)$  of a diagram of an oriented link is the sum of the signs of the crossings of  $D$ , where each crossing has  $+1$  or  $-1$  as defined before in Figure 6.

**Definition 2.8.** Addition of two oriented knots  $K_1$  and  $K_2$  denoted by  $K_1 + K_2$ , is defined as follows: Consider  $K_1$  and  $K_2$  in different copies of  $S^3$ , say  $K_1 \subset S_1$  and  $K_2 \subset S_2$ . Remove from each of  $S_1$  and  $S_2$  a ball  $B_1$  and  $B_2$  respectively, both of which are 3 – balls such that  $B_1 \cap K_1$  and  $B_2 \cap K_2$  are unknotted arcs. Identify  $\partial B_1$  and  $\partial B_2$  such that  $\partial B_1 \cap K_1$  and  $\partial B_2 \cap K_2$  are identified to each other in such a way that orientations match properly.

**Definition 2.9.** A knot  $K$  is said to be prime knot if it is not the unknot, and if  $K = K_1 + K_2$  it implies that  $K_1$  or  $K_2$  is the unknot.

**Definition 2.10.** The crossing number of a knot is the minimum of the number of crossings taken over all the diagrams of the knot.

**Definition 2.11.** Unknotting number  $u(K)$  of a knot is the minimum number of crossing exchanges needed to change  $K$  to the unknot.

Every knot has an unknotting number. This can be seen as follows:  
 choose some point  $x$  on the knot, choose a direction and move along the knot in that direction starting from  $x$ . As you move along the knot and reach a crossing first time, make it a over-pass if it is an under-pass.If it is an under-pass leave it unchanged. Make no changes when you reach a crossing a second time. Stop this process after going around the knot once. After this process you will get a section of the knot on which we can apply type I reidemeister move to simplify the diagram. This whole process reduces the number of crossings in the diagram atleast by one and thus repeating this process for the rest of the crossings will give you the unknot. The figure below illustrates the process for an arbitrary knot.

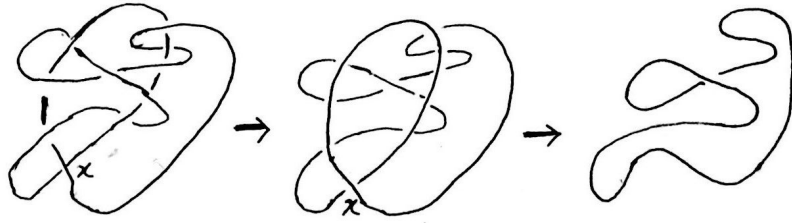


Figure 7: algorithm to unknot a knot by crossing changes

**Definition 2.12.** A knot is called *alternating* if in the diagram of the knot, the crossings alternate between over and under as we travel along the knot.

### Dowker and Thistlethwaite coding for knots

This is a method using which we assign to every knot a unique sequence of pairs of integer upto reflection. Choose a base point and move along the knot in a direction and allocate each crossing a positive integer  $1, 2, \dots$  in the order they come as we move along the knot. Therefore each crossing gets two numbers assigned to it, one from an over pass and one from an under pass, one odd and one even integer. Suppose at a crossing both the underpass and over-pass are both assigned even or odd numbers, it would mean that there are odd number of crossings in the portion of the knot bound by this crossing. But if some other portion of the link enters this region bounded by the portion at a crossing then it also has to exit the region, i.e., each section entering adds two crossings to the section. Therefore the total number of crossings in the section is even. This is a contradiction and hence our assumption is wrong and every crossing has one odd number and one even number assigned to it.

Thus a diagram with  $n$ -crossings gives a pairing between the first  $n$  odd and even numbers. Now we put a minus sign in front of the even number if it is allocated to an underpass or else it has no sign in front of it. If the knot (neglecting orientation) is prime, we can write it uniquely using the above method. There are finitely many ways to describe a knot, we choose the lowest possible  $n$  and then the first description in a lexicographical ordering of the strings of even numbers is called the canonical name for the knot.

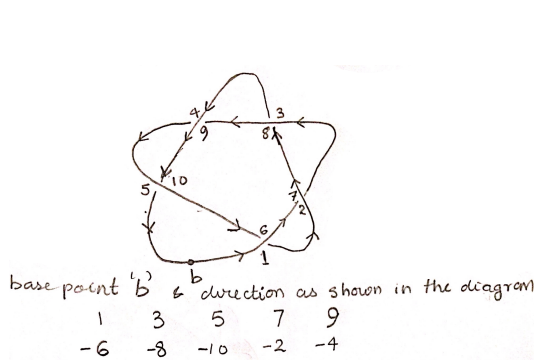


Figure 8:

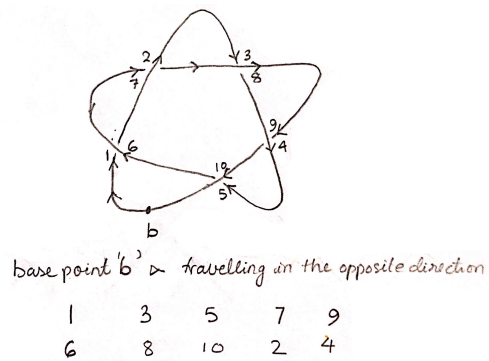


Figure 9:

**Definition 2.13.** Consider a two component link  $L$  with oriented components  $L_1$  and  $L_2$ , then link number  $lk(L_1, L_2)$  of  $L_1$  and  $L_2$  is half the sum of the signs of the crossings at which one strand is from  $L_1$  and other is from  $L_2$ .

## Different kinds of knots and links

### Pretzel link

**Definition 2.14.** A Pretzel link  $P(a_1, \dots, a_n)$  is a link with  $n$  tassels and  $a_i$  denoting the number of crossings in the  $i^{\text{th}}$  tassel.

If a tassel has a right handed twist then  $a_i$  is positive and if it has opposite sense of twisting then  $a_i$  is a negative integer.

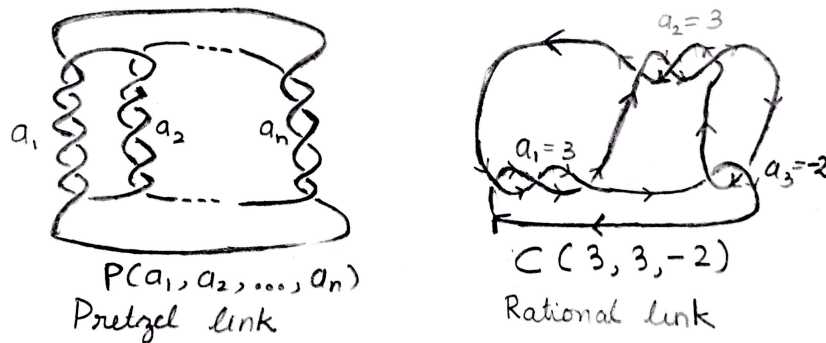


Figure 10: Pretzel links

A Rational link  $C(a_1, \dots, a_n)$  has at most two components. A rational link is a  $(p, q)$  rational link if

$$\frac{q}{p} = \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}$$

For the example shown in the Figure 10,  $(p, q) = (17, 5)$ .

### Braids

**Definition 2.15.** A braid of  $n$  strings is defined in the following way: Consider  $n$  oriented arcs traversing a rectangle from left to right which join  $n$  fixed points on the left to same number of points on the right. These arcs cannot turn back as they go from left to right i.e. any vertical line drawn inside the rectangle intersects each of the arcs at exactly one point.

Two braids are said to be the same if they are ambient isotopic.

**Definition 2.16.** Let  $b$  be some braid. Join the end points from the outside of the box in a standard way, i.e. join the first point on the left end to the first point on the right end, second on the left to second on the right and so on. This is defined to be the closure of a braid  $\hat{b}$ . See figure 11

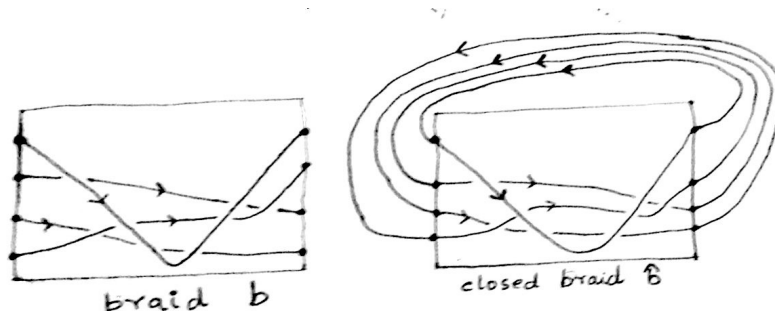


Figure 11: Braid and its closure

### Satellite knot

From any given knot  $K$ , we can construct a more complicated knot by the process described below.

- i) Take a knot  $K$  in a solid torus  $T$ . We call this the *pattern*.
- ii) Now consider  $e : T \rightarrow S^3$ , an embedding such that  $eT$  is a regular neighbourhood of a knot  $C$  in  $S^3$ . Then  $eK$  is called the satellite of  $C$  and  $C$  is called the companion of  $eK$ .
- iii) We can make many knots from given  $K$  and  $C$  by just twisting  $T$  as it embeds around  $C$ . Shown below in the figure 12 is the construction of a satellite of the trefoil knot.

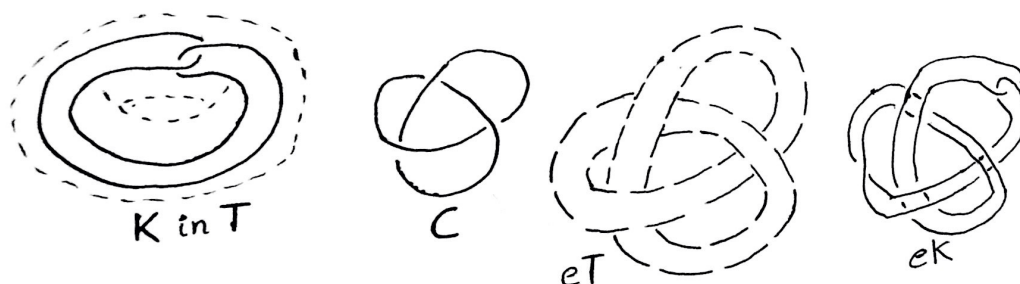


Figure 12: Satellite knot

## 3 Seifert Surfaces and Knot factorisation

### Seifert surface and graph

**Definition 3.1.** A connected compact oriented surface in  $S^3$  which has oriented link  $L$  as its oriented boundary is called a Seifert surface of the link  $L$ .

**Theorem 3.2.** Any oriented link in  $S^3$  has a Seifert surface.

*Proof.* : Let  $D$  be an oriented diagram of an oriented link. Then by smoothing out the crossings by a process called *splicing* according to the orientations of the link as in Figure 13 we construct a new diagram which is denoted as  $\hat{D}$ . Thus  $\hat{D}$  differs from  $D$  only at the crossings. Now, these form simple, disjoint, oriented curves which are closed and therefore bound discs in  $S^3$ . Join two such discs with half-twisted strips at all the crossings as shown in Figure 14. Thus we have constructed an oriented surface which has  $L$  as the boundary. Each disc is given the orientation from the orientation of  $\hat{D}$ . Now if this surface is not connected, we make them connected by removing small discs and inserting long thin tubes. □

**Definition 3.3.** From given Seifert surfaces of a knot (or a link) we construct a graph by shrinking each of the discs to a vertex and shrinking each of the twisted bands along their breadth to an edge. This graph is defined to be the Seifert graph of a knot.

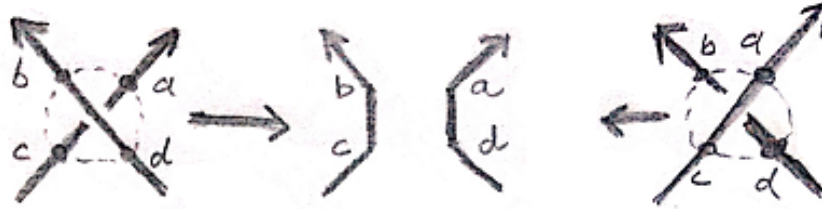


Figure 13: Splicing of a knot at a crossing

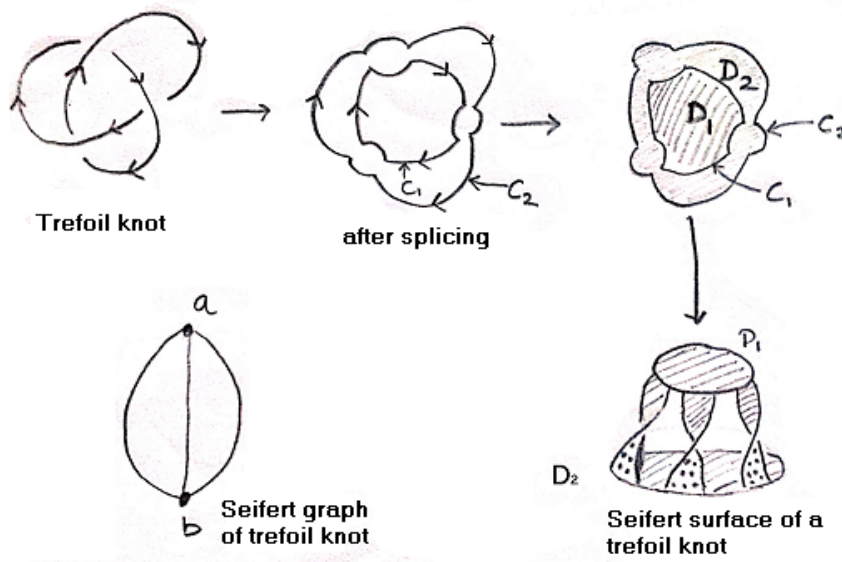


Figure 14: Seifert surface and graph of Trefoil knot

## Genus of a knot

**Theorem 3.4.** Any orientable closed surface  $F$  can be decomposed into discs. If such a decomposition has  $\alpha_0$  points,  $\alpha_1$  edges and  $\alpha_2$  faces then Euler characteristic of  $F$  is given by

$$\chi(F) = \alpha_0 - \alpha_1 + \alpha_2;$$

The Euler characteristic does not depend on how the surface is decomposed. The genus  $\mathbf{g}(F)$  and  $\chi(F)$  are related by the following equation:

$$\chi(F) = 2 - 2\mathbf{g}(F) \tag{1}$$

If  $F$  has boundary then

$$\chi(F) = 2 - \mu(F) - 2\mathbf{g}(F) \tag{2}$$

where  $\mu(F)$  is the number of the boundary components of  $F$ .

There is a really easy way to calculate the Euler characteristic of a Seifert surface using the method by which these surfaces are constructed in the proof of Theorem 2.2. The points in  $F$  are the four vertices of the bands. The edges of  $F$  are the curves which constitute the boundary of the bands and the boundaries between the vertices. The faces of  $F$  are the discs and the bands. If  $b$  is the number of bands,  $d$  is the number of discs then  $\alpha_0 = 4b$ ,  $\alpha_1 = 6b$  and  $\alpha_2 = b + d$ .

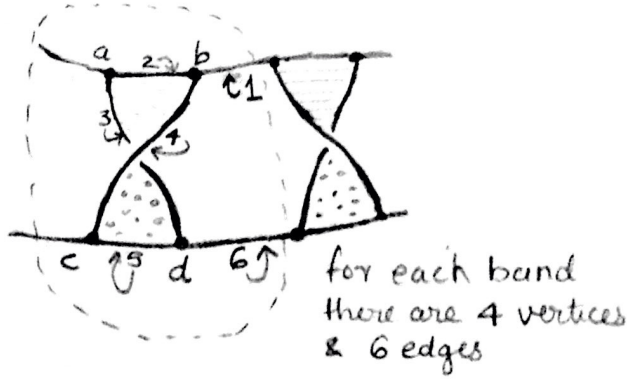


Figure 15:

Hence, Euler characteristic of  $F$

$$\chi(F) = 4b - 6b + (b + d) = d - b$$

and the genus  $\mathbf{g}(F)$  is

$$\begin{aligned} 2\mathbf{g}(F) &= 2 - \mu(K) - \chi(F) \\ &= 2 - \mu(K) - b + d \end{aligned}$$

$$2\mathbf{g}(F) + \mu(K) - 1 = 1 - b + d \tag{3}$$

When  $K$  is a knot,  $2\mathbf{g}(F) = 1 - d + b$ .

**Definition 3.5.** The genus  $\mathbf{g}(K)$  of a knot  $K$  is defined as minimum of genus of all Seifert surfaces for  $K$ .

**Theorem 3.6.** For any two knots  $K_1$  and  $K_2$ ,

$$\mathbf{g}(K_1 + K_2) = \mathbf{g}(K_1) + \mathbf{g}(K_2).$$

*Proof.* Let  $K_1$  and  $K_2$  be knots and  $F_1$  and  $F_2$  their Seifert surfaces respectively in  $S^3$ . We choose an arc  $\alpha$  from a point in  $K_1$  to a point in  $K_2$  that does not meet  $F_1 \cup F_2$  at any other point in its interior. It also intersects a 2-sphere separating  $K_1$  from  $K_2$  at one point. Consider a thin strip around  $\alpha$  which joins  $F_1 \cup F_2$ , this is the Seifert surface of  $K_1 + K_2$ . Thus  $\mathbf{g}(K_1 + K_2) \leq \mathbf{g}(K_1) + \mathbf{g}(K_2)$ .

Let  $F$  be minimal Seifert surface for  $K_1 + K_2$ ,  $\Sigma$  be a 2-sphere which intersects  $K_1 + K_2$  transversely at two points. Thus  $K_1 + K_2$  is separated into two arcs  $\alpha_1$  and  $\alpha_2$ . Also, there exists an arc  $\beta$  which joins the two points in the intersection such that  $\alpha_1 \cup \beta$  and  $\alpha_2 \cup \beta$  are  $K_1$  and  $K_2$  respectively. Now, both  $F$  and  $\Sigma$  are surfaces in  $S^3$ , therefore are a sub-complex of some triangulation of  $S^3$ .

We isotope  $\Sigma$  so that it is transverse to  $F$ .  $F \cap \Sigma$  is then a 1-manifold which is a finite collection of simple closed curves and also contains  $\beta$ . Consider a closed curve  $C$  which is innermost on  $\Sigma \setminus \beta$ . This means  $C$  bounds a disc  $D$  in  $\Sigma$  whose interior does not intersect  $F$  at all. Do a surgery on  $F$  using this  $D$  by creating a surface  $\hat{F}$  as follows. Remove an annular neighbourhood of  $C$  and replace it by two discs which are copies of  $D$  one on either side of  $D$ . If  $C$  did not separate  $F$ ,  $\hat{F}$  would have lower genus than  $F$ . But this is not possible, hence  $\hat{F}$  is disconnected. The component of  $\hat{F}$  that contains  $K_1 + K_2$  is a surface with same genus as  $F$ , but meet  $\Sigma$  at fewer number of closed curves. Repeating this process gives a surface  $F'$  of same genus as  $F$  that intersects  $\Sigma$  only at  $\beta$ . Thus  $\Sigma$  separates  $F'$  into two surface which are Seifert surfaces for  $K_1$  and  $K_2$ . Hence  $g(K_1) + g(K_2) \leq g(K_1 + K_2) = g(F')$ . Therefore,  $g(K_1) + g(K_2) = g(K_1 + K_2)$ .  $\square$

**Corollary 3.7.** *No non-trivial knot has an additive inverse. If  $K_1 + K_2$  is the unknot then each of  $K_1$  and  $K_2$  is an unknot.*

**Corollary 3.8.** *If  $K$  is a non-trivial knot and let  $\sum_1^n K$  denote the sum of  $n$  copies of  $K$ , then  $\sum_1^n K \neq \sum_1^m K$  if  $n \neq m$ . This way we can create infinitely many distinct knots.*

**Corollary 3.9.** *Any knot of genus 1 is a prime knot.*

**Corollary 3.10.** *A knot can be written as a finite sum of prime knots.*

**Theorem 3.11.** (Schönflies Theorem). *Any piecewise linear embedding  $e : S^2 \rightarrow S^3$  divides  $S^3$  into two components, the closure of each of which is a piecewise linear ball.*

Using this we prove the following theorem and also the theorem which gives the uniqueness of the prime decomposition of knots.

**Theorem 3.12.** *If some knot  $K$  can be written as  $K = P + Q$ , where  $P$  is a prime knot and  $K$  can also be written as  $K = K_1 + K_2$ , Then one of the following is true:*

- (i)  $K_1 = P + K'_1$  for some  $K'_1$  and  $Q = K'_1 + K_2$ , or
- (ii)  $K_2 = P + K'_2$  for some  $K'_2$  and  $Q = K_1 + K'_2$ .

**Corollary 3.13.** *If  $P$  is a prime knot and  $P + Q = K_1 + K_2$  and  $P = K_1$  then  $Q = K_2$ .*

*Proof.* By Theorem 2.12, there are two possibilities, either  $P + K'_1 = K_1 = P$  and  $Q = K'_1 + K_2$  for some  $K'_1$  or  $K_2 = P + K'_2$  and  $Q = K'_2 + K_1$  for some  $K'_2$ . In the first case genus of  $K'_1$  must be zero, so it is the unknot and  $Q = K_2$ , and in the second case  $Q = K'_2 + P = K_2$ .  $\square$

**Theorem 3.14.** *There exists a unique expression for a given knot  $K$  as finite sum of prime knots. This uniqueness is upto ordering of summands.*

*Proof.* Suppose  $K = P_1 + P_2 + \dots + P_m = Q_1 + Q_2 + \dots + Q_n$ , where  $P_i$  and  $Q_j$  are all prime knots. By the previous theorem,  $P_1$  is a summand of one of the  $Q_j$  or  $Q_2 + Q_3 + \dots + Q_n$ , and if it is the latter then it is the summand of one of the  $Q_j$  by induction on  $n$ . If  $P_1$  is summand of a  $Q_j$ ,  $P_1 = Q_j$ . Thus we can cancel them from both sides of the equation. Follow this process as an induction on  $m$  to get the result, where induction starts at  $m = 0$ . Then  $n = 0$  because the unknot cannot be expressed as sum of non-trivial knots by genus considerations.  $\square$

## 4 Kauffman bracket and Jones Polynomials

**Definition 4.1.** *Kauffman bracket is a function from unoriented link diagrams in the oriented plane to Laurent polynomials with integer coefficients in an indeterminate  $A$ . The image of a link diagram  $D$  is  $\langle D \rangle \in \mathbb{Z}[A^{-1}, A]$  such that:*

- (i)  $\langle \bigcirc \rangle = 1$ , i.e. the bracket of the unknot is the constant polynomial 1.
- (ii)  $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^2)\langle D \rangle$ ,
- (iii)  $\langle \times \rangle = A\langle \nearrow \rangle + A^{-1}\langle \searrow \rangle$ .

**Lemma 4.2.** *The change in the bracket polynomial of a diagram by a Type I Reidemeister move is:*

- (i)  $\langle \overbrace{\curvearrowright} \rangle = -A^3\langle \underbrace{\curvearrowleft} \rangle$
- (ii)  $\langle \overbrace{\curvearrowleft} \rangle = -A^{-3}\langle \underbrace{\curvearrowright} \rangle$ .

*Proof.* (i)

$$\begin{aligned} \langle \overbrace{\curvearrowright} \rangle &= A\langle \underbrace{\curvearrowright} \rangle + A^{-1}\langle \overbrace{\curvearrowleft} \rangle \\ &= (A(-A^{-2} - A^2) + A^{-1})\langle \underbrace{\curvearrowleft} \rangle \\ &= -A^3\langle \underbrace{\curvearrowleft} \rangle \end{aligned}$$

(ii)

$$\begin{aligned} \langle \overbrace{\curvearrowleft} \rangle &= A\langle \overbrace{\curvearrowright} \rangle + A^{-1}\langle \underbrace{\curvearrowright} \rangle \\ &= (A + A^{-1}(-A^{-2} - A^2))\langle \underbrace{\curvearrowleft} \rangle \\ &= -A^{-3}\langle \underbrace{\curvearrowleft} \rangle \end{aligned}$$

□

**Lemma 4.3.** *A Type II or Type III Reidemeister move does not change the bracket polynomial  $\langle D \rangle$  of diagram  $D$ .*

$$(i) \langle \overbrace{\curvearrowright} \rangle = \langle \underbrace{\curvearrowright} \rangle, \quad (ii) \langle \overbrace{\curvearrowleft} \rangle = \langle \underbrace{\curvearrowleft} \rangle.$$

$\langle D \rangle$  is invariant under regular isotopy of  $D$ .

*Proof.* (i)

$$\begin{aligned} \langle \overbrace{\curvearrowright} \rangle &= A\langle \overbrace{\curvearrowright} \rangle + A^{-1}\langle \overbrace{\curvearrowleft} \rangle \\ &= A(-A^{-3}\langle \overbrace{\curvearrowright} \rangle) + A^{-1}(A\langle \overbrace{\curvearrowleft} \rangle + A^{-1}\langle \overbrace{\curvearrowright} \rangle) \\ &= -A^{-2}\langle \overbrace{\curvearrowright} \rangle + \langle \overbrace{\curvearrowleft} \rangle + A^{-2}\langle \overbrace{\curvearrowright} \rangle \\ &= \langle \underbrace{\curvearrowright} \rangle \end{aligned}$$

(ii)

$$\begin{aligned} \langle \overbrace{\curvearrowleft} \rangle &= A\langle \underbrace{\curvearrowleft} \rangle + A^{-1}\langle \overbrace{\curvearrowright} \rangle \\ &= A\langle \underbrace{\curvearrowleft} \rangle + A^{-1}\langle \overbrace{\curvearrowright} \rangle \\ &= \langle \underbrace{\curvearrowleft} \rangle \end{aligned}$$

□

**Remark:** The writhe of  $D$ ,  $w(D)$ , changes by  $+1$  or  $-1$  when a diagram  $D$  changes by Type I Reidemeister move. The writhe of an oriented link and bracket polynomial of the diagram without the orientation are invariant under regular isotopy and the change in them under Type I Reidemeister move is known.

**Theorem 4.4.** *Let  $D$  be a diagram of an oriented link  $L$ . Then the expression*

$$(-A)^{-3w(D)}\langle D \rangle$$

*is an invariant of the oriented link  $L$ .*

*Proof.* From Lemma 3.3 we see that the above expression is unchanged by Type II and Type III Reidemeister moves. By Lemma 3.2 and the discussion about  $w(D)$  above we see that it is unchanged by a Type I move. We know as well that any two equivalent diagrams of a link differ by a sequence of these moves. So we can conclude that it is an invariant of the link  $L$ .  $\square$

**Definition 4.5.** *The the Jones polynomial  $V(L)$  of an oriented link  $L$  is the Laurent polynomial in  $t^{1/2}$ , defined as*

$$V(L) = \left( (-A)^{-3w(D)}\langle D \rangle \right)_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$$

*where  $D$  is oriented diagram for  $L$ .*

For a knot, the Jones polynomial always has only integer powers of  $t$ . Also  $V(\bigcirc) = 1$ . The Jones polynomial for a knot does not depend on the orientation of the knot because at each crossing the orientation of both strands change and hence the sign does not change.

For an oriented link  $L$  with every component having orientation, the polynomial for a link  $L^*$  constructed from  $L$  by changing the orientation of just one component is given by  $V(L^*) = t^{-3lk(K, L \setminus K)}V(L)$  because the writhe changes by  $-lk(K, L \setminus K)$ .

**Proposition 4.6.** *The Jones polynomial invariant is a function*

$$V : \{\text{oriented links in } S^3\} \longrightarrow \mathbb{Z}[t^{-1/2}, t^{1/2}]$$

*such that:*

(i)  $V(\text{unknot}) = 1$

(ii) *whenever three oriented links  $L_+$ ,  $L_-$  and  $L_0$  differ only at neighbourhood of a crossing as shown in the Figure 16, then*

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0$$

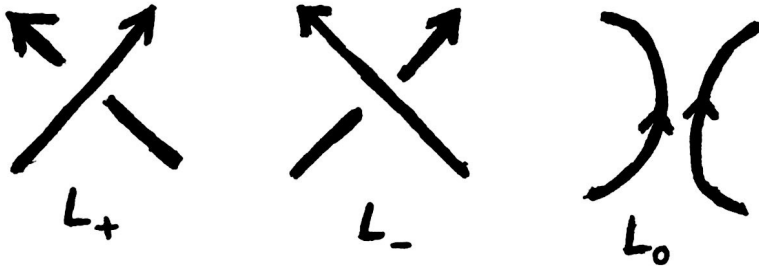


Figure 16: .

*Proof.*

$$\begin{aligned}\langle \times \rangle &= A \langle \rangle \langle \rangle + A^{-1} \langle \rangle \langle \rangle \\ \langle \times \rangle &= A \langle \rangle \langle \rangle + A^{-1} \langle \rangle \langle \rangle\end{aligned}$$

Multiplying the first equation by  $A$ , second by  $A^{-1}$  and subtracting them gives

$$A \langle \times \rangle - A^{-1} \langle \times \rangle = (A^2 - A^{-2}) \langle \rangle \langle \rangle$$

Using the fact that in those links with diagrams as shown in Figure 16,  $w(L_+) - 1 = w(L_0) = w(L_-) + 1$

$$-A^4 V(L_+) + A^{-4} V(L_-) = (A^2 - A^{-2}) V(L_0)$$

Substitute  $t^{\frac{1}{2}} = A^{-2}$  to get the required relation. □

The Jones polynomial for  $K_1 + K_2$ , where  $K_1$  and  $K_2$  are knots, is  $V(K_1 + K_2) = V(K_1)V(K_2)$ . Though the same formula works for links but summation of two links is not well-defined as the second link can be added through any of its components to any of the components of the first link. Thus two distinct links can have the same Jones polynomial. If  $\bar{L}$  is the reflection of a link  $L$ , then  $w(D) = -w(\bar{D})$  hence  $V(\bar{L})$  is obtained from  $V(L)$  by interchanging  $t^{-1/2}$  and  $t^{1/2}$ .

Two distinct knots can also have same Jones polynomial. Consider the Kinoshita-Terasaka knot and the Conway knot (see Figure 18 ). They are two distinct knots which are related by a process called mutation. But these two have same Jones polynomial.

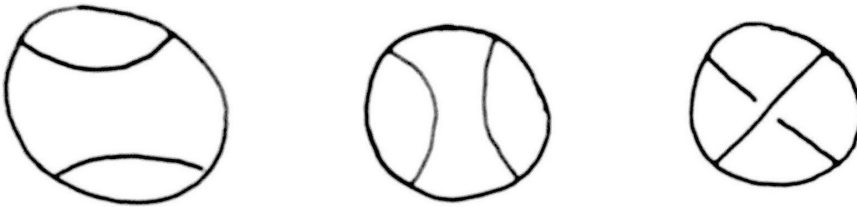


Figure 17:

Mutation of a knot is done as follows:

Assume there exists a ball in  $S^3$  whose boundary intersects the knot at four points. Remove this ball, rotate it by  $\pi$  about an axis such that these four points on the boundary are preserved and then replace the ball in  $S^3$  to get a possibly new knot. The possible axes of rotation are the following: one perpendicular to the plane of the diagram, the north-south axis and east-west axis. For oriented knots, we have to change the orientation the strands inside the ball along with the rotation. This is called mutation of the knot. Figure 18 shows a mutation of Kinoshita-Terasaka knot into a Conway knot.

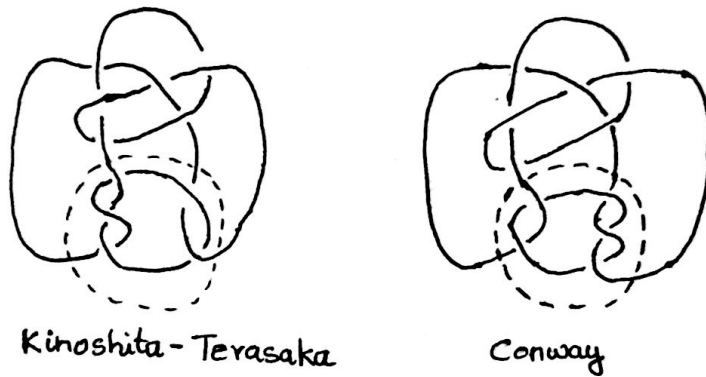


Figure 18: Mutation

Now to calculate the Jones polynomial of these knots we first calculate it for the crossings in the ball using the Proposition 3.6 after reducing all the crossings to exactly one of the forms in the Figure 17 so that the rotation does not change the calculations. Then we calculate it for the rest of the knot and multiply both the polynomials to get the Jones polynomial of the mutated knot as  $V(K_1 + K_2) = V(K_1)V(K_2)$  which proves that mutation doesn't change the Jones polynomial.

### Calculating the Jones polynomial for a trefoil knot



Figure 19: Trefoil knot

For calculating the Jones polynomial, we first calculate the bracket polynomial, then the writhe of the knot and then we calculate the Jones polynomial.

#### Step 1. The bracket polynomial.

$$\begin{aligned}
 \langle \text{Trefoil} \rangle &= A \langle \text{Crossing 1} \rangle + A^{-1} \langle \text{Crossing 2} \rangle \\
 &= A(-A^3 \langle \text{Circle} \rangle) + A^{-1}(-A^{-3} \langle \text{Circle} \rangle) \\
 &= -A^4 - A^{-4} \\
 \langle \text{Trefoil} \rangle &= A \langle \text{Crossing 1} \rangle + A^{-1} \langle \text{Crossing 2} \rangle \\
 &= A \langle \text{Crossing 1} \rangle + A^{-1}(-A^{-3} \langle \text{Crossing 3} \rangle) \\
 &= A(-A^4 - A^{-4}) + A^{-1}(-A^{-3}(-A^{-3} \langle \text{Circle} \rangle)) \\
 &= A^{-7} - A^{-3} - A^5
 \end{aligned}$$

Step 2. The writhe of  $w(\text{Trefoil}) = +3$ .

**Step 3. Use the Definition 3.5.**

$$\begin{aligned}
 V(\text{link}) &= \left( (-A)^{-3w(D)} \langle D \rangle \right)_{t^{1/2}=A^{-2}} \\
 &= \left( (-A)^{-3(3)} (A^{-7} - A^{-3} - A^5) \right)_{t^{1/2}=A^{-2}} \\
 &= \left( (-A)^{-9} (A^{-7} - A^{-3} - A^5) \right)_{t^{1/2}=A^{-2}} \\
 &= \left( (-A^{-16} + A^{-12} + A^{-4}) \right)_{t^{1/2}=A^{-2}} \\
 &= -t^4 + t^3 + t
 \end{aligned}$$

## 5 Alexander Polynomials via Seifert matrices

### Seifert matrices

The Seifert graph  $\Gamma(D)$  of a link diagram  $D$  is always planar dividing  $S^2$  into  $f$  faces or domains. So using notations as in section 2

$$2 = \chi(S^2) = d - b + f.$$

Hence,

$$f - 1 = 1 - d + b = 2\mathbf{g}(F) + \mu(K) - 1$$

and in case of knots  $f - 1 = 2\mathbf{g}(F)$ . We can consider  $S^2$  as  $\mathbb{R}^2$  with an additional point at infinity.  $2\mathbf{g}(F) + \mu(K) - 1$  is equal to the number of domains (excluding the one which contains infinity) in  $S^2$  bound by  $\Gamma(D)$  and hence is equal to the number of domains bounded by  $\Gamma(D)$  in  $\mathbb{R}^2$ . Boundary of each of these domains is a closed curve and we can use these closed curves to create certain closed curves  $\alpha_i$ 's on the Seifert surface. The closed curves are as shown in the diagram below.

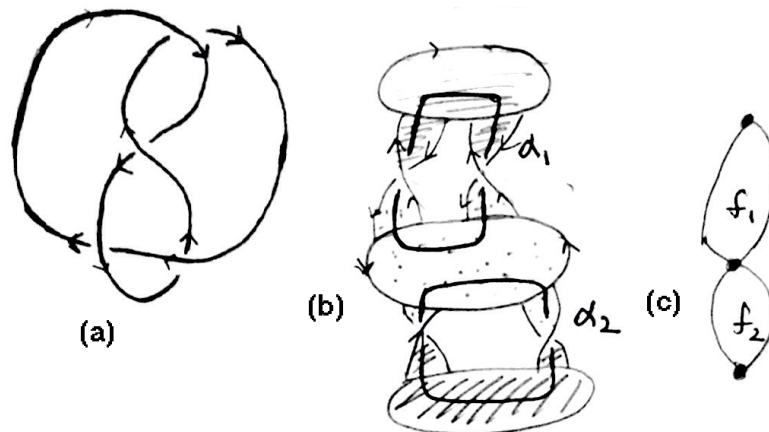


Figure 20:  $4_1$

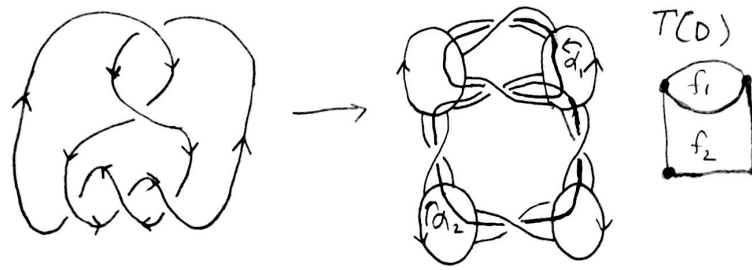


Figure 21:  $5_2$

Sometimes these closed curves  $\alpha_i$  and  $\alpha_j$  may intersect each other and thus they may not form a link. We avoid these intersections by the below described processes:

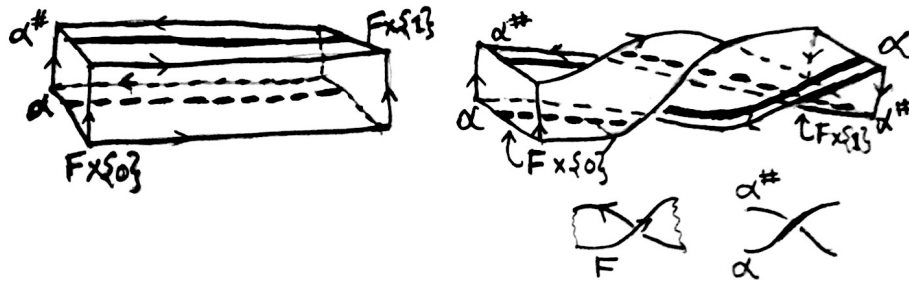


Figure 22:

Suppose  $\{\alpha_i, \alpha_j\}$  do not form a link then we can lift both of them slightly above the surface so as to get new curves  $\alpha_i^\#$  and  $\alpha_j^\#$  such that  $\{\alpha_i, \alpha_i^\#\}$  and  $\{\alpha_j, \alpha_j^\#\}$  are links. If we assign an orientation to  $\alpha_i, \alpha_j$  then it will induce naturally some orientation on  $\alpha_i^\#, \alpha_j^\#$ . Now, we can calculate the linking number  $lk(\alpha_i, \alpha_i^\#)$ ,  $lk(\alpha_i, \alpha_j^\#)$ ,  $lk(\alpha_j, \alpha_i^\#)$  and  $lk(\alpha_j, \alpha_j^\#)$ .

**Definition 5.1.** The  $m \times m$  matrix  $M$ , where  $m$  is the number of  $\alpha_i$

$$M = [lk(\alpha_i, \alpha_j^\#)]_{i,j=1,2,3,\dots,m}$$

is defined as the Seifert matrix of a knot  $K$ .

**Example** Calculation of Seifert matrix for a right-handed trefoil knot.

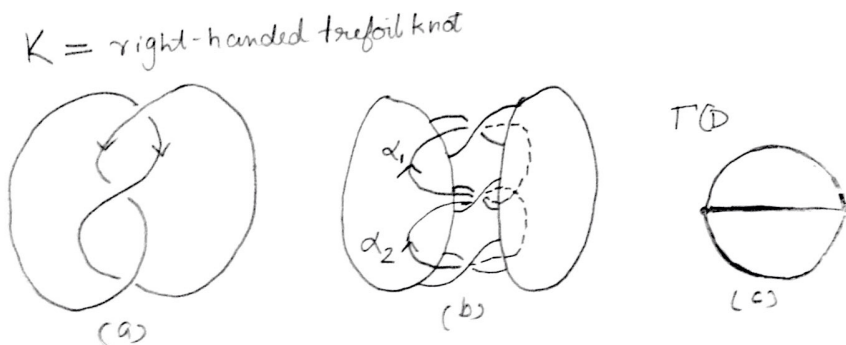


Figure 23:  $3_1$

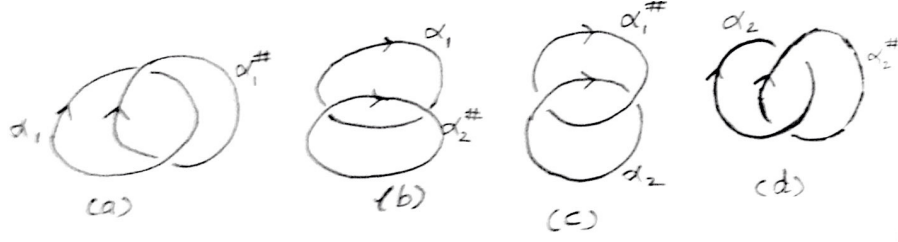


Figure 24:

From the above diagrams we see that

$$lk(\alpha_1, \alpha_1^\#) = -1, lk(\alpha_2, \alpha_1^\#) = 1, lk(\alpha_2, \alpha_2^\#) = -1, lk(\alpha_1, \alpha_2^\#) = 0$$

Therefore Seifert matrix

$$M = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

## S-equivalence of Seifert Matrices

Here I have included some theorems without proof regarding the  $S$ -equivalence of Seifert matrices.

**Theorem 5.2.** *Two Seifert matrices calculated from two equivalent knots or links can be changed from one to another by applying a finite number of times the following operations,  $\Lambda_1$  and  $\Lambda_2$ , and their inverses:*

$$\Lambda_1 : M_1 \rightarrow PM_1P^T,$$

where  $P$  is an invertible integer matrix with  $\det P = \pm 1$ .

$$\Lambda_2 : M_1 \rightarrow M_2 = \begin{bmatrix} & * & 0 \\ & \vdots & \vdots \\ M_1 & & \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} & 0 & 0 \\ & \vdots & \vdots \\ M_1 & & \\ * & \cdots & * & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

where  $*$  denotes some arbitrary integer.

**Definition 5.3.** *If a matrix  $M$  is obtained from another matrix  $M'$  by applying finite number of  $\Lambda_1$ ,  $\Lambda_2$  and the inverse  $\Lambda_2^{-1}$ , then  $M$  and  $M'$  are said to be  $S$ -equivalent and it is denoted by  $M \stackrel{S}{\sim} M'$ .*

**Theorem 5.4.** *Let  $D$  be a regular diagram of a knot and  $D'$  be the regular diagram obtained after a single Reidemeister move on it. Let  $F$  is a Seifert diagram for  $D$  and  $F'$  for  $D'$  such that  $F$  and  $F'$  differ at only the regions affected by the Reidemeister moves. Then the Seifert matrices  $M$  and  $M'$  obtained from  $F$  and  $F'$ , respectively, are  $S$ -equivalent.*

**Theorem 5.5.** *Suppose  $M_1$  and  $M_2$  are two Seifert matrices obtained from Seifert surfaces  $F_1$  and  $F_2$  of a knot  $K$ . Then  $M_1$  and  $M_2$  are  $S$ -equivalent.*

**Theorem 5.6.** *Suppose  $K$  is an oriented knot and  $rK$  is the knot with the reverse orientation, Then  $M_{rK} \stackrel{S}{\sim} M_K^T$ .*

**Theorem 5.7.** *Suppose  $K$  is an oriented knot and let  $\rho K$  be its reflection, then  $M_{\rho K} \stackrel{S}{\sim} -M_K^T$ .*

## The Alexander Polynomial

We will now use the machinery developed to find a knot invariant.

**Definition 5.8.** Let  $M$  be a Seifert matrix of a knot  $K$ , then  $|\det(M + M^T)|$  is called the determinant of  $K$ .

**Proposition 5.9.** The determinant of  $K$  is an invariant of knot  $K$ .

(It is enough to show that the determinant does not change value if we apply  $\Lambda_1^\pm$  and  $\Lambda_2^\pm$ .)

**Proposition 5.10.**  $\det(M - M^T) = 1$  where  $M$  is a Seifert matrix of a knot  $K$ .

**Definition 5.11.** If  $M$  is a Seifert matrix of a knot (or link)  $K$  and its order is  $k$ , then the Alexander polynomial of  $K$  denoted by  $\Delta_K(t)$  is

$$\Delta_K(t) = t^{-\frac{k}{2}} \det(M - tM^T)$$

Here  $k = 2g + \mu(F) - 1$  because  $k$  is the number of faces bounded by Seifert graph in  $\mathbb{R}^2$ . When  $K$  is a knot,  $k = 2g(F)$  and hence  $\Delta_K(t)$  is a Laurent polynomial with terms as integers powers of  $t$ .

**Theorem 5.12.** Suppose  $M_1$  and  $M_2$  are the Seifert matrices of a knot  $K$ , and  $r$  and  $s$  are the orders of  $M_1$  and  $M_2$  respectively, then :

$$t^{-\frac{r}{2}} \det(M_1 - tM_1^T) = t^{-\frac{s}{2}} \det(M_2 - tM_2^T)$$

This shows that The Alexander polynomial is an invariant of the knot  $K$ .

**Theorem 5.13.** Suppose  $K$  is a knot, then  $\Delta_K(t)$  is a symmetric Laurent polynomial,

$$\Delta_K(t) = a_{-n}t^{-n} + a_{-(n-1)}t^{-(n-1)} + \dots + a_{n-1}t^{n-1} + a_n t^n$$

and

$$a_n = a_{-n}, a_{-(n-1)} = a_{n-1}, \dots, a_{-1} = a_1.$$

*Proof.*  $M$  be a Seifert matrix of  $K$  and  $k$  is the order of  $M$ . Since  $K$  is a knot,  $k$  is even.

$$\begin{aligned} \Delta_K(t^{-1}) &= t^{-\frac{k}{2}} \det(M - t^{-1}M^T) \\ &= t^{-\frac{k}{2}} \det(tM - M^T) \\ &= (-1)^k t^{-\frac{k}{2}} \det(M^T - tM) \\ &= t^{-\frac{k}{2}} \det(M - tM^T)^T \\ &= t^{-\frac{k}{2}} \det(M - tM^T) = \Delta_K(t^1) \end{aligned}$$

□

**Proposition 5.14.**  $|\Delta_K(-1)|$  is equal to the determinant of a knot  $K$ .

**Proposition 5.15.** If  $K$  is the unknot, then  $\Delta_K(t) = 1$ .

## The Alexander-Conway polynomial

This polynomial was constructed by John Conway so as to make the calculation of Alexander polynomial easier. It is constructed using the *skein diagrams* as shown in the Figure 25.

**Definition 5.16.** Given an oriented knot  $K$ , we assign an Laurent polynomial,  $\nabla_K(z)$ , with a fixed indeterminate  $z$ , using the following axioms:

**Axiom 1:** If  $K$  is the unknot, then we assign  $\nabla_K(z) = 1$ .

**Axiom 2:** Suppose  $D_+, D_-, D_0$  are the regular diagram of three knots  $K_+, K_-, K_0$  such that they are same everywhere except for a neighbourhood of one crossing. They differ from each other as shown in the following figure.

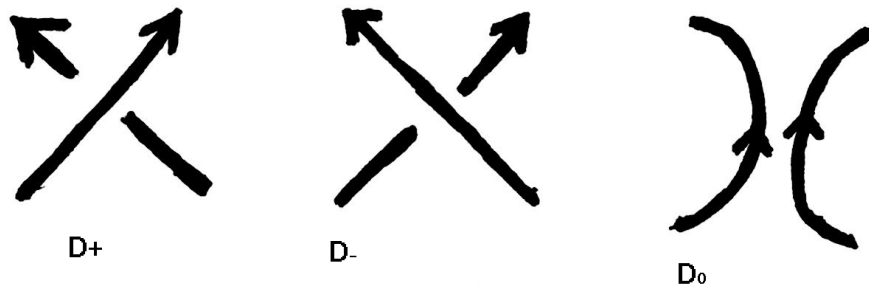


Figure 25:

Then the Laurent Polynomials of the three are related as follows

$$\nabla_{K_+}(z) - \nabla_{K_-}(z) = z\nabla_{K_0}(z).$$

**Theorem 5.17.**  $\Delta_K(t) = \nabla_K(\sqrt{t} - \frac{1}{\sqrt{t}})$ .

**Example** For a trefoil knot

$$\nabla_K(z) = 1\nabla_O(z) + z\nabla_{OO}(z) + z^2\nabla_O(z)$$

but  $\nabla_O(z) = 1$  and  $\nabla_{OO}(z) = 0$ , i.e.  $\Delta_K(t) = \nabla_K(\sqrt{t} - \frac{1}{\sqrt{t}})$

$$\Delta_K(t) = 1 + (\sqrt{t} - \frac{1}{\sqrt{t}})^2 = t^{-1} - 1 + t.$$

## 6 Conclusion

In this project I learnt introductory aspects of knot theory, different types of knots, some knot invariants and how to calculate them. I studied the construction of Seifert surfaces from knots, calculate the genus and Euler characteristic, construct the Seifert matrices etc..The two knot invariants- the Jones polynomial and Alexander polynomial which were covered in this project. The methods of calculating them as discussed in the report, the Jones polynomial using the Kauffman bracket polynomial and the Alexander polynomial of knots using two different methods, one using the Seifert matrix and other one was using the Alexander-Conway polynomial, are some of the few ways to calculate these invariants. A more advanced project in knot theory as a continuation of this project can include the other advanced methods to calculate the above invariants, learning to calculate the knot group, the other polynomial invariants attached to knots, studying the knots using homology theory etc..

## References

1. W.B.R. Lickorish; An Introduction to Knot Theory
2. Kunio Murasugi; Knot Theory and its Applications
3. [en.wikipedia.org/wiki/Knot theory](https://en.wikipedia.org/wiki/Knot_theory)